LESSON

# 2 Graphs

A function can be fully described by showing what happens at each number in its domain (for example, $4 \rightarrow 2$) or by giving its formula (for example, $f(x) = \sqrt{x}$). However, neither of these provides a clear overall picture of the function.

Luckily for us, René Descartes came up with the idea of a *graph*, a visual picture of a function. Rather than say $4 \rightarrow 2$ or $f(4) = 2$, we plot (4,2) on the Cartesian plane, which would look like Figure 2.1.

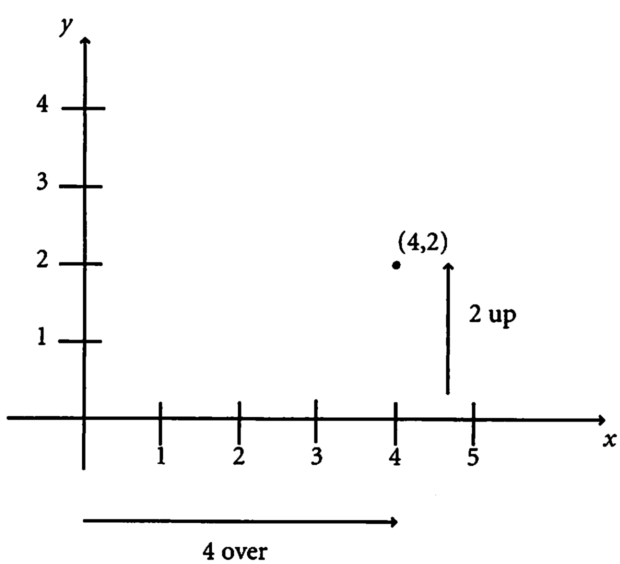


**Figure 2.1**

## Note on Finding Coordinates

We put the $y$ into the formula $y = f(x) = \sqrt{x}$ to imply that the $y$-coordinates of our points are the numbers we get by plugging the $x$-coordinates into the function $f$.

## ▶ Practice

Plot the following points on a Cartesian plane.

**1.** (3,5)
**2.** (−3,4)
**3.** (2,−6)
**4.** (−1,−5)
**5.** (0,3)
**6.** (−5,0)
**7.** (0,0)
**8.** $\left(4\frac{1}{2},\frac{1}{4}\right)$

For the function $f(x) = x^2 - 2x + 5$, plot the point at the following positions.

**9.** $x = 3$
**10.** $x = 1$
**11.** $x = 0$
**12.** $x = -2$

If we plotted *all* the points in the domain of $f(x) = \sqrt{x}$ (not just the whole numbers, but all the fractions and decimals, too), then the points would be so close together that they would form a continuous curve as in Figure 2.2.

The graph shows us several interesting characteristics of the function $f(x) = \sqrt{x}$. Because the graph starts at $x = 0$ and runs to the right, this means that the domain is $x \geq 0$.

We can see that the function $f(x) = \sqrt{x}$ is *increasing* (going up from left to right) and not *decreasing* (going down from left to right).

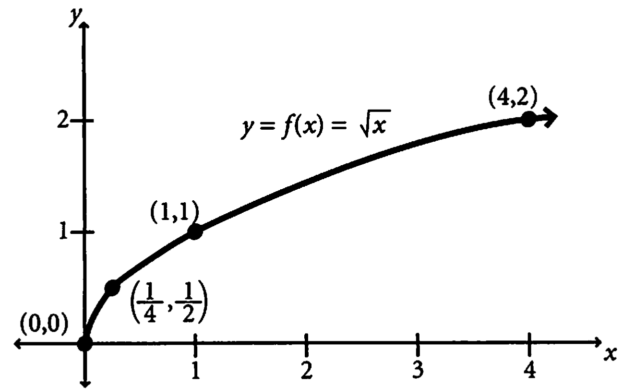


**Figure 2.2**

The function $f(x) = \sqrt{x}$ is *concave down* because it curves downward (see Figure 2.3) like a frown and not *concave up* like a smile (see Figure 2.4).

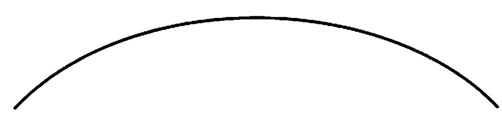

**Figure 2.3**

**Figure 2.4**

### Example

Use the graph of the following function (see Figure 2.5) to determine the domain, where the function is increasing and decreasing, and where the function is concave up and concave down.

## Mathematical Notation Note

**An apology must be made for mathematical notation here. An expression like (2,8) is ambiguous. Is this a single point with coordinates $x = 2$ and $y = 8$? Is this an interval consisting of all the points between 2 and 8? Only the context can make clear which is meant. If we read "at (2,8)," then this is a single point. If we read "on (2,8)," then it refers to an interval.**

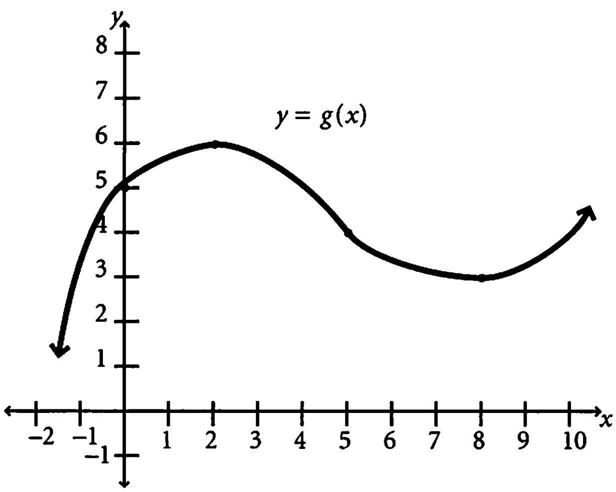


**Figure 2.5**

### Solution

The domain of $g$ consists of all real numbers because there is a point above or below every number on the $x$-axis.

The function $g$ is increasing up to the point at $x = 2$, where it then decreases down to $x = 8$, and then increases ever afterward. To save space, we say that $g$ increases on $(-\infty,2)$ and on $(8,\infty)$, and that $g$ decreases on (2,8).

The point at (2,6) where $g$ stops increasing and begins to decrease is the highest point in its immediate area and is called a *local maximum*. The point at (8,3) is similarly a *local minimum*, the lowest point in its neighborhood. These points tend to be the most interesting points on a graph.

The concavity of $g$ is trickier to estimate. Clearly $g$ is concave down in the vicinity of $x = 2$ and concave up around $x = 7$ and $x = 8$. The exact point where the concavity changes is called a *point of inflection*. On this graph, it seems to be at the point (5,4), though some people might imagine it a bit earlier or later. Thus, we say that $g$ is concave down on $(-\infty,5)$ and concave up on $(5,\infty)$.

To be completely honest, any information obtained by looking at a graph is going to be a rough estimate. Is the local maximum at (2,6), or is it at (2.0003,5.9998)? There is no way to tell the difference. Graphs made up by people, like the ones in this lesson, tend to have everything interesting happen at whole numbers. Graphs formed using real-world data tend to be much less kind.

### Example

Use the graph in Figure 2.6 to identify the domain of $h$, where it is increasing and decreasing, where it has local maxima and minima, where it is concave up and down, and where it has points of inflection.

### Solution

The first thing to notice is that $h$ has three breaks, or *discontinuities*. If we wanted to trace the graph of $h$ with a continuous motion of a pencil, then we would have to lift up the pencil at $x = -2$, $x = 2$, and at $x = 5$. The little circle at (5,3) indicates a hole in the graph where a single point has been taken out. This means that $x = 5$ is not in the domain, just as $x = -1$ has no point above or below it. The situation at $x = 2$ is more interesting because $x = 2$ *is* in the domain,

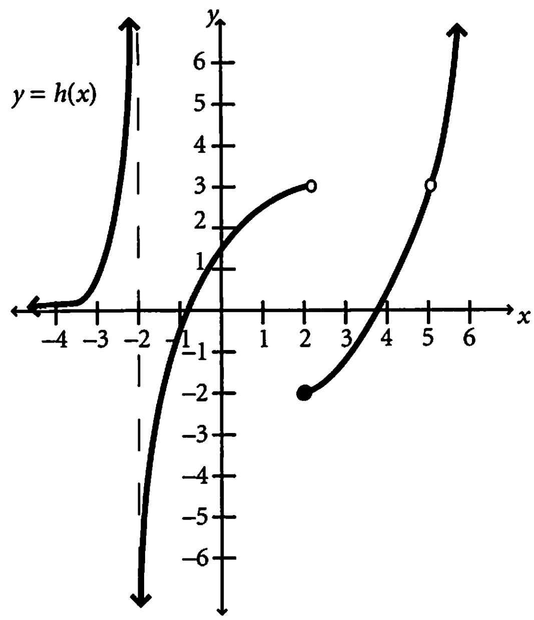


**Figure 2.6**

with the point (the shaded-in circle) at $(2,-2)$ representing $h(2) = -2$. All of the points immediately before $x = 2$ have $y$-values close to $y = 3$, but then there is an abrupt jump down to $x = 2$. Such jumps look awkward on a graph, but occur often in real life, like the way the cost of postage leaps up as soon as a letter weighs more than one ounce.

Because of the discontinuities, we have to name each interval separately, as in: $h$ increases on $(-\infty,-2)$, $(-2,2)$, $(2,5)$, and on $(5,\infty)$. As well, $h$ is concave up on $(-\infty,-2)$, $(2,5)$, and on $(5,\infty)$, and concave down on $(-2,2)$.

There is a local minimum at $(2,-2)$, because the point there is the lowest in its immediate vicinity, $1 < x < 3$. There is no local maximum in that range because the $y$-values get really close to $y = 3$; there is no highest point in the range.

Similarly, a point of inflection can be seen at $x = 2$ but not at $x = -2$ because there can't be a point of inflection where there is no point!

The situation at $x = -2$ is called an *asymptote* because the graph begins to flatten out like a straight line. The more we would continue to draw the graph off the top and bottom, the straighter this line would become. In this case, $x = -2$ is a *vertical asymptote* because it approximates a vertical line at $x = -2$. Because the graph appears to flatten out like the straight horizontal line $y = 0$ (the $x$-axis) as the graph goes off to the left, this means that the graph of $y = h(x)$ appears to have a *horizontal asymptote* at $y = 0$.

## ▶ Practice

Use the graph of each function to determine the domain, the discontinuities, where the function is increasing and decreasing, the local maximum and minimum points, where the function is concave up and down, the points of inflection, and the asymptotes.

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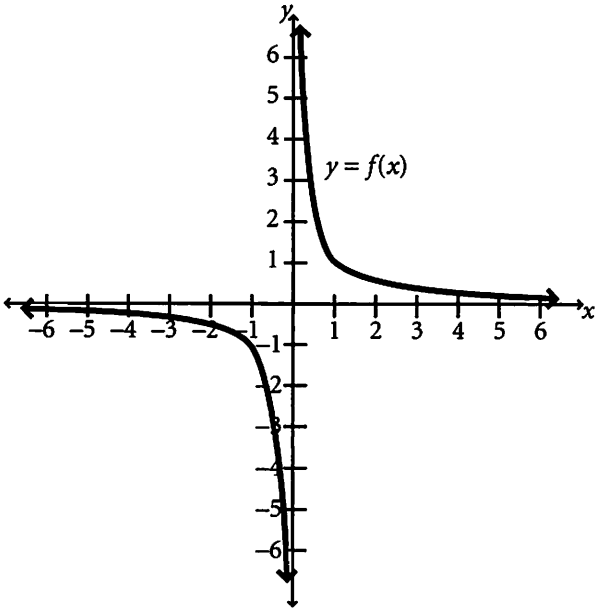


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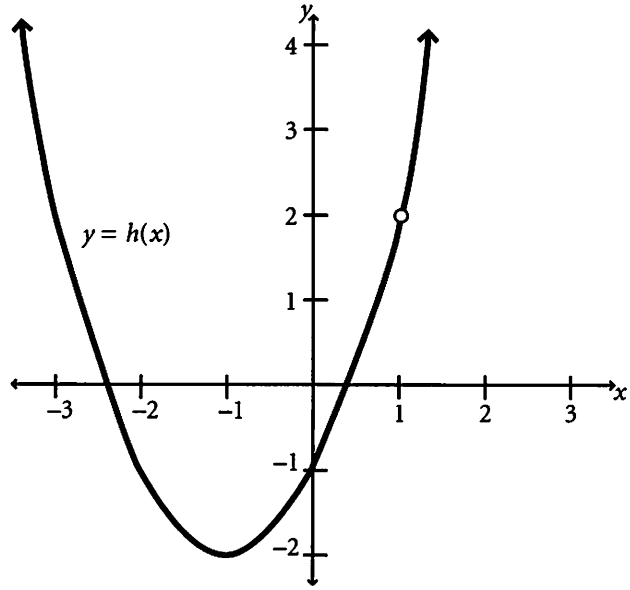


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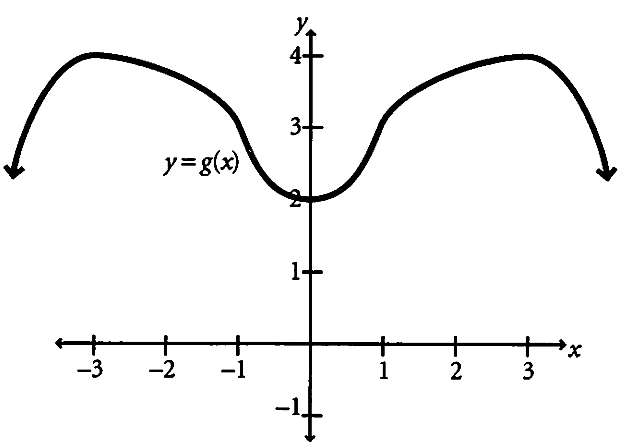


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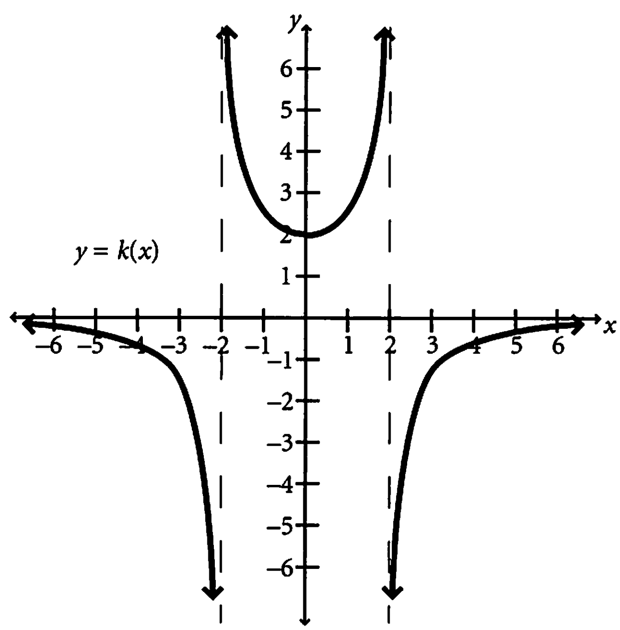

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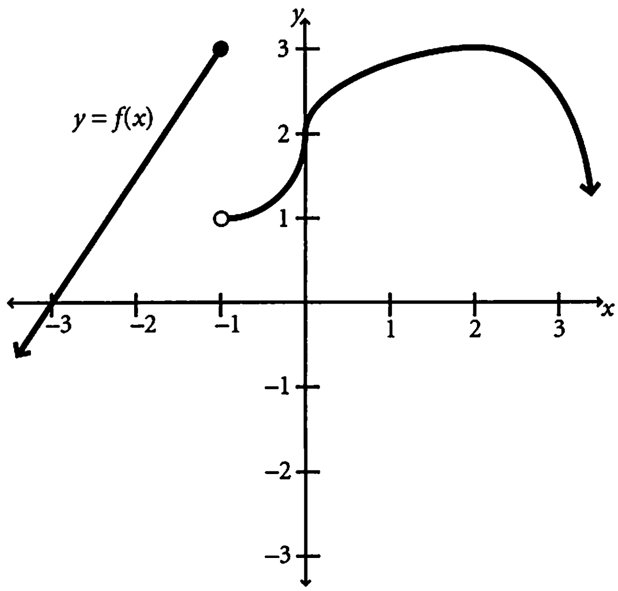


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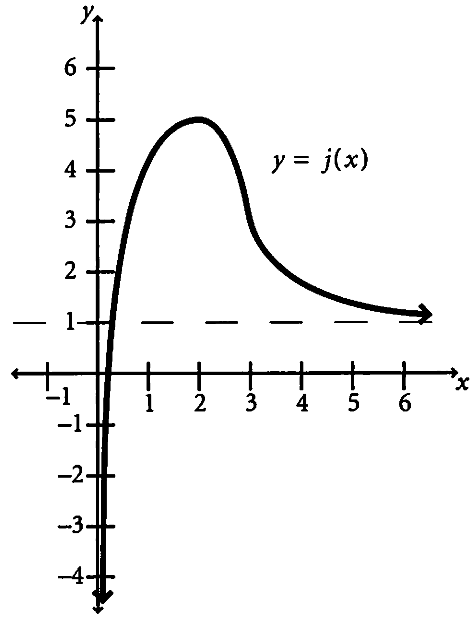


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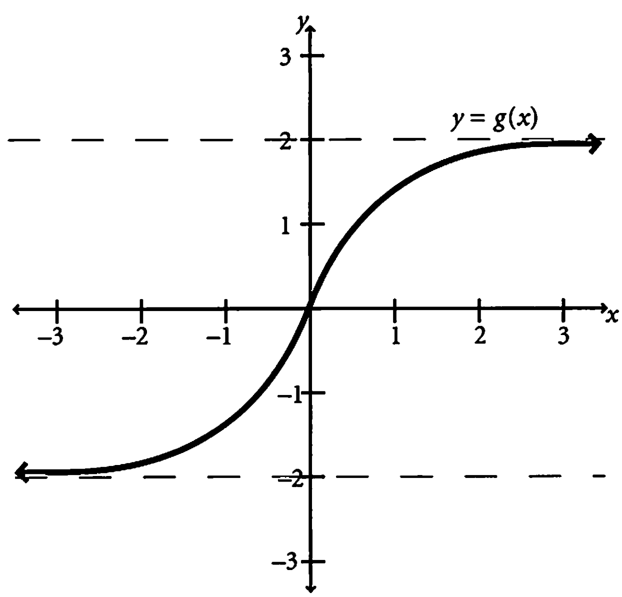


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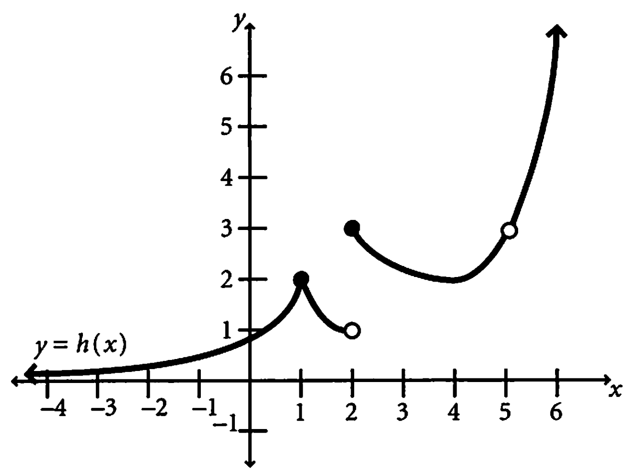


## ▶ Note

We can obtain all sorts of useful information from a graph, such as its maximal points, where it is increasing and decreasing, and so on. Calculus will enable us to get this information directly from the function. We will then be able to draw graphs intelligently, without having to calculate and plot thousands of points (the method graphing calculators use).

## ▶ Straight Lines

The easiest and most beloved of all graphs are straight lines. Human beings tend to build, move, and even think in straight lines. There is something calming and reassuring about straight lines. With any two points, we can immediately tell how much a line is increasing or decreasing, as seen in Figure 2.7.

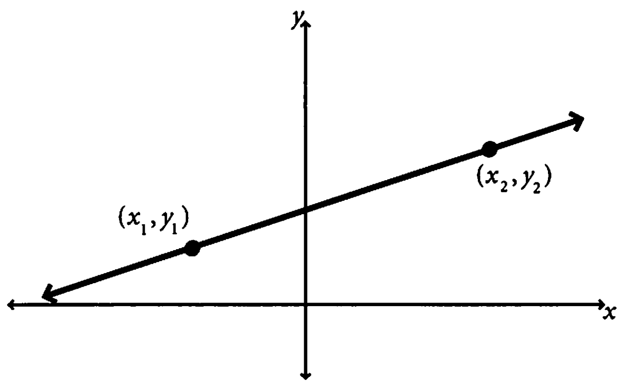


**Figure 2.7**

"How much a line is increasing or decreasing" is called the *slope* and is calculated by dividing "rise over run":

$$slope = \frac{rise}{run}$$

$$= \frac{y\text{-}change}{x\text{-}change}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

### Example

What is the slope of the line through points (2,7) and (−1,5)?

### Solution

$$\text{slope} = \frac{5 - 7}{-1 - 2} = \frac{-2}{-3} = \frac{2}{3}$$

## ▶ Practice

Find the slope between the following points.

**21.** (1,5) and (2,8)
**22.** (2,5) and (6,7)
**23.** (7,3) and (−2,3)
**24.** (−2,−4) and (−6,5)
**25.** (2,7) and (5,$w$)
**26.** (4,10) and ($x$,$y$)

## ▶ Point-Slope Formula

The most wonderful thing about straight lines is that their slopes are always the same. Thus, if a straight line has slope $m$ and goes through the point $(x_1, y_1)$, then any other point $(x,y)$ on the line will calculate the same slope:

$$\frac{y - y_1}{x - x_1} = m$$

By cross-multiplying, we get the *point-slope formula* for finding the equation of a straight line:

$$y - y_1 = m(x - x_1)$$

or equivalently

$$y = m(x - x_1) + y_1$$

Here, $y$ is a function of $x$, which could be written as

$$y(x) = m(x - x_1) + y_1.$$

### Example

Find the equation of the line with slope −2 through point (−1,8). Graph the line.

### Solution

$y = -2(x - (-1)) + 8$

$y = -2x + 6$

This form of the equation is called the *slope-intercept* form because $-2$ is the slope and 6 is where the line intercepts the $y$-axis (see Figure 2.8):

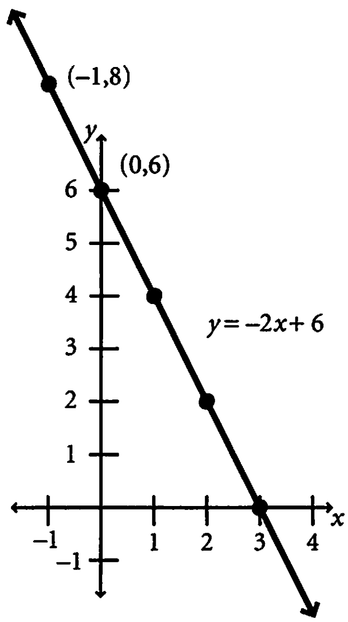


**Figure 2.8**

The slope of $-2 = \frac{-2}{1}$ means the $y$-value goes down 2 with every 1 increase in the $x$-value.

### Example

Find the equation of the straight line through (2,6) and (5,7). Graph the line.

### Solution

The slope is $\frac{7-6}{5-2} = \frac{1}{3}$, so the equation is $y = \frac{1}{3}(x - 2) + 6 = \frac{1}{3}x + \frac{16}{3}$ (see Figure 2.9).

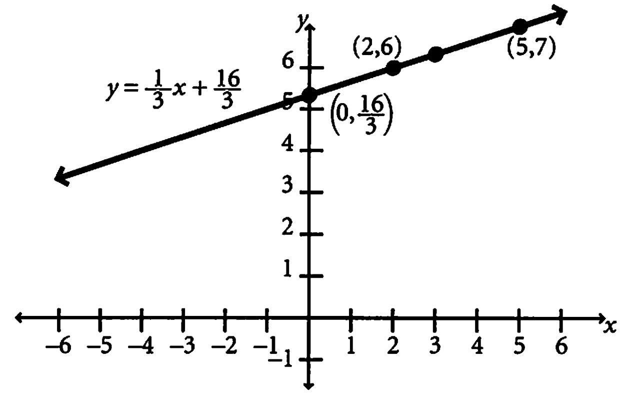


**Figure 2.9**

The slope of $\frac{1}{3}$ means the $y$-value goes up 1 when the $x$-value increases by 3.

## ▶ Practice

Find the equation of the straight line with the given information and then graph the line.

**27.** slope 2 through point $(1,-2)$

**28.** slope $-\frac{2}{3}$ through point $(6,1)$

**29.** through points $(5,3)$ and $(-1,-3)$

**30.** through points $(2,5)$ and $(6,5)$